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ON THE SYMMETRICAL FORM OF THE DIFFERENTIAL EQUATIONS OF PLANETARY MOTIONS.

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The differential equations usually employed in determining the motions of the planets,

$$\begin{aligned}\frac{d^2x_i}{dt^2} + \frac{k^2(1+m_i)}{r_i^3} x_i &= \frac{\partial R_i}{\partial x_i}, \\ \frac{d^2y_i}{dt^2} + \frac{k^2(1+m_i)}{r_i^3} y_i &= \frac{\partial R_i}{\partial y_i}, \quad i = 1, 2, \dots, n. \quad (1) \\ \frac{d^2z_i}{dt^2} + \frac{k^2(1+m_i)}{r_i^3} z_i &= \frac{\partial R_i}{\partial z_i},\end{aligned}$$

are unsymmetrical, since the functions R_1, R_2, \dots, R_n are not the same for each planet. The symmetrical form* may be obtained in the following manner :—

1. Let ξ_i, η_i, ζ_i be the coordinates of the different masses of the system M_i referred to fixed rectangular axes. Let G_i be the center of gravity of the masses M_1, M_2, \dots, M_i ; let x_i, y_i, z_i be the coordinates of M_i referred to three axes parallel to the fixed axes, but passing through G_{i-1} ; let X_i, Y_i, Z_i be the coordinates of G_i ; $\mu_i = \sum_{\sigma=1}^i m_{\sigma}$. If the number of the bodies be n , G_n will be the center of gravity of the system. We shall also have

$$\xi_i = X_{i-1} + x_i, \quad (2)$$

in which $X_1 = \xi_1$; and, if we put $x_1 = 0$, we have $X_0 = \xi_1$, in which X_0 is introduced merely in order that the nomenclature in (2) may be applicable throughout.

We have, also, in accordance with the properties of centers of gravity,

$$\mu_{i-1}X_{i-1} + m_i\xi_i = \mu_iX_i; \quad (3)$$

whence, substituting for ξ_i its value as given by (2), we obtain

$$\begin{aligned}\mu_iX_{i-1} + m_ix_i &= \mu_iX_i, \\ m_ix_i &= \mu_i(X_i - X_{i-1}).\end{aligned} \quad (4)$$

* See Tisserand's *Mécanique Céleste*, t. i, chap. iv, the substance of which is derived from an interesting memoir by M. R. Radau, entitled " Sur une transformation des équations différentielles de la Dynamique " (*Annales de l'École Normale*, 1^{re} série, t. v).

Squaring (2) and multiplying by m_i , we have

$$m_i \xi_i^2 = m_i x_i^2 + 2m_i x_i X_{i-1} + m_i X_{i-1}^2;$$

whence, adding $m_i x_i (X_i - X_{i-1}) - \frac{m_i^2}{\mu_i} x_i^2 = 0$,

$$m_i \xi_i^2 = m_i \frac{\mu_{i-1}}{\mu_i} x_i^2 + m_i x_i (X_i + X_{i-1}) + m_i X_{i-1}^2.$$

If $\mu_i (X_i - X_{i-1})$ be substituted for $m_i x_i$ in the middle term, this becomes

$$m_i \xi_i^2 = m_i \frac{\mu_{i-1}}{\mu_i} x_i^2 + \mu_i (X_i^2 - X_{i-1}^2) + m_i X_{i-1}^2$$

$$= m_i \frac{\mu_{i-1}}{\mu_i} x_i^2 + \mu_i X_i^2 - \mu_{i-1} X_{i-1}^2;$$

or, since $\mu_0 = 0$,

$$\sum_1^n m_i \xi_i^2 = \sum_1^n m_i \frac{\mu_{i-1}}{\mu_i} x_i^2 + \mu_n X_n^2. \quad (5)$$

2. Equation (5) may be obtained in a still simpler manner. Equation (4) gives

$$X_i = X_{i-1} + \frac{m_i}{\mu_i} x_i;$$

whence, substituting in (2), we have

$$\xi_i = X_i + \frac{\mu_{i-1}}{\mu_i} x_i. \quad (6)$$

From (3) we have, also,

$$m_i \xi_i = \mu_i X_i - \mu_{i-1} X_{i-1}. \quad (7)$$

Multiplying the left hand member by ξ_i , and the terms of the right hand member by the values of ξ_i given by (6) and (2), respectively, equation (7) becomes

$$\begin{aligned} m_i \xi_i^2 &= \mu_i X_i^2 - \mu_{i-1} X_{i-1}^2 + \mu_{i-1} x_i (X_i - X_{i-1}) \\ &= \mu_i X_i^2 - \mu_{i-1} X_{i-1}^2 + m_i \frac{\mu_{i-1}}{\mu_i} x_i^2; \end{aligned}$$

whence, as before,

$$\sum_1^n m_i \xi_i^2 = \sum_1^n m_i \frac{\mu_{i-1}}{\mu_i} x_i^2 + \mu_n X_n^2. \quad (5)$$

3. If we differentiate (2) and (3) with reference to t , we see at once that the relations between the differentials are exactly the same as the relations

between the corresponding variables ; hence we may substitute the differentials for the variables in (5), and obtain

$$\sum_1^n m_i \left[\frac{d\xi_i}{dt} \right]^2 = \sum_1^n m_i \frac{\mu_{i-1}}{\mu_i} \left[\frac{dx_i}{dt} \right]^2 + \mu_n \left[\frac{dX_n}{dt} \right]^2. \quad (8)$$

There are also relations similar to (5) and (8) for the coordinates η and ζ . Adding, we have

$$\sum_1^n m_i \rho_i^2 = \sum_1^n m_i \frac{\mu_{i-1}}{\mu_i} r_i^2 + \mu_n R_n^2, \quad (9)$$

in which $\rho_i^2 = \xi_i^2 + \eta_i^2 + \zeta_i^2$, $r_i^2 = x_i^2 + y_i^2 + z_i^2$, and $R_n^2 = X_n^2 + Y_n^2 + Z_n^2$; also

$$\begin{aligned} 2T &\equiv \sum_1^n m_i \left[\left[\frac{d\xi_i}{dt} \right]^2 + \left[\frac{d\eta_i}{dt} \right]^2 + \left[\frac{d\zeta_i}{dt} \right]^2 \right] \\ &= \sum_1^n m_i \frac{\mu_{i-1}}{\mu_i} \left[\left[\frac{dx_i}{dt} \right]^2 + \left[\frac{dy_i}{dt} \right]^2 + \left[\frac{dz_i}{dt} \right]^2 \right] \\ &\quad + \mu_n \left[\left[\frac{dX_n}{dt} \right]^2 + \left[\frac{dY_n}{dt} \right]^2 + \left[\frac{dZ_n}{dt} \right]^2 \right]. \end{aligned} \quad (10)$$

4. An examination of (6) and (2) shows that we may write

$$\frac{d\eta_i}{dt} = \frac{dY_i}{dt} + \frac{\mu_{i-1}}{\mu_i} \frac{dy_i}{dt} = \frac{dY_{i-1}}{dt} + \frac{dy_i}{dt}. \quad (11)$$

Multiplying $m_i \xi_i$ by the first member of (11), $\mu_i X_i$ by the second, and $\mu_{i-1} X_{i-1}$ by the third, equation (7) becomes

$$\begin{aligned} m_i \xi_i \frac{d\eta_i}{dt} &= \mu_i X_i \frac{dY_i}{dt} + \mu_{i-1} X_i \frac{dy_i}{dt} - \mu_{i-1} X_{i-1} \frac{dY_{i-1}}{dt} - \mu_{i-1} X_{i-1} \frac{dy_i}{dt} \\ &= \mu_i X_i \frac{dY_i}{dt} - \mu_{i-1} X_{i-1} \frac{dY_{i-1}}{dt} + m_i \frac{\mu_{i-1}}{\mu_i} x_i \frac{dy_i}{dt}. \end{aligned}$$

A similar expression can be readily obtained for $m_i \eta_i \frac{d\xi_i}{dt}$; whence

$$\begin{aligned} \sum_1^n m_i \left[\xi_i \frac{d\eta_i}{dt} - \eta_i \frac{d\xi_i}{dt} \right] &= \mu_n \left[X_n \frac{dY_n}{dt} - Y_n \frac{dX_n}{dt} \right] \\ &\quad + \sum_1^n m_i \frac{\mu_{i-1}}{\mu_i} \left[x_i \frac{dy_i}{dt} - y_i \frac{dx_i}{dt} \right]. \end{aligned} \quad (12)$$

5. If U be the force function of the system, we may put

$$P = \frac{\partial T}{\partial \frac{dX_n}{dt}}, \quad P_1 = \frac{\partial T}{\partial \frac{dY}{dt}}, \quad P_2 = \frac{\partial T}{\partial \frac{dZ}{dt}};$$

$$p_{3i} = \frac{\partial T}{\partial \frac{dx_i}{dt}}, \quad p_{3i+1} = \frac{\partial T}{\partial \frac{dy_i}{dt}}, \quad p_{3i+2} = \frac{\partial T}{\partial \frac{dz_i}{dt}};$$

and write the equations of motion in the well-known canonical form,

$$\begin{aligned} \frac{dX_n}{dt} &= \frac{\partial(T - U)}{\partial P}, \dots; & \frac{dx_i}{dt} &= \frac{\partial(T - U)}{\partial p_{3i}}, \dots; \\ \frac{dP}{dt} &= -\frac{\partial(T - U)}{\partial X_n}, \dots; & \frac{dp_{3i}}{dt} &= -\frac{\partial(T - U)}{\partial x_i}, \dots \end{aligned} \quad (13)$$

Since U does not contain X_n , Y_n , Z_n , but only the differences $\xi_i - \xi_j$, $\eta_i - \eta_j$, $\zeta_i - \zeta_j$, equations (10) and (13) give

$$\frac{d^2 X_n}{dt^2} = \frac{d^2 Y_n}{dt^2} = \frac{d^2 Z_n}{dt^2} = 0;$$

whence, by integration,

$$X_n = at + a', \quad Y_n = \beta t + \beta', \quad Z_n = \gamma t + \gamma', \quad (14)$$

in which a , a' , β , β' , γ , γ' are arbitrary constants.

Also, since T does not contain the x_i , y_i , z_i , but only the velocities, (10) and (13) give

$$m_i \frac{\mu_{i-1}}{\mu_i} \frac{d^2 x_i}{dt^2} = \frac{\partial U}{\partial x_i}, \quad m_i \frac{\mu_{i-1}}{\mu_i} \frac{d^2 y_i}{dt^2} = \frac{\partial U}{\partial y_i}, \quad m_i \frac{\mu_{i-1}}{\mu_i} \frac{d^2 z_i}{dt^2} = \frac{\partial U}{\partial z_i}, \quad (15)$$

which have the symmetrical form desired.